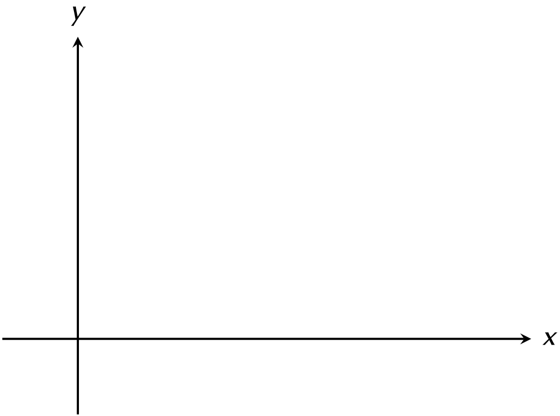
Derivatives

Philipp Warode

October 1, 2019

- **Goal:** Quantify the "variation" of a function
  How much do the y-values change if the input x is varied?



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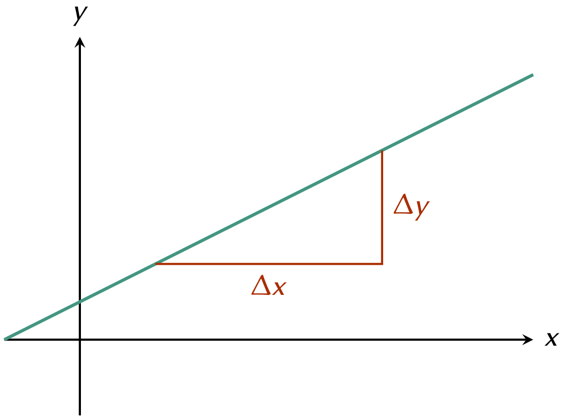

- The slope of the line is $m = \dfrac{\Delta y}{\Delta x}$

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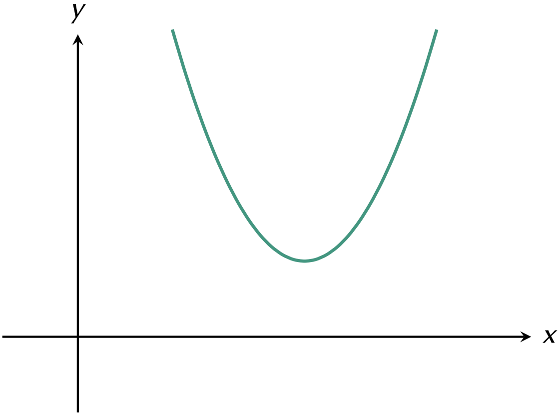

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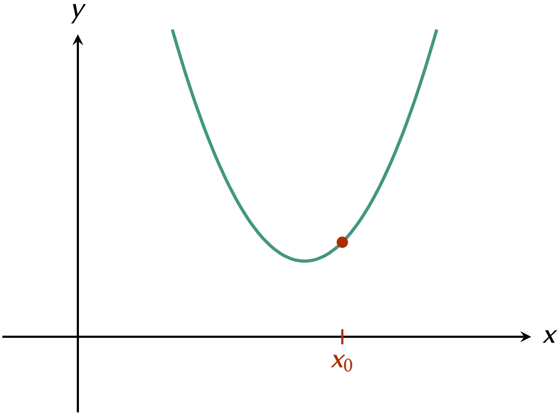

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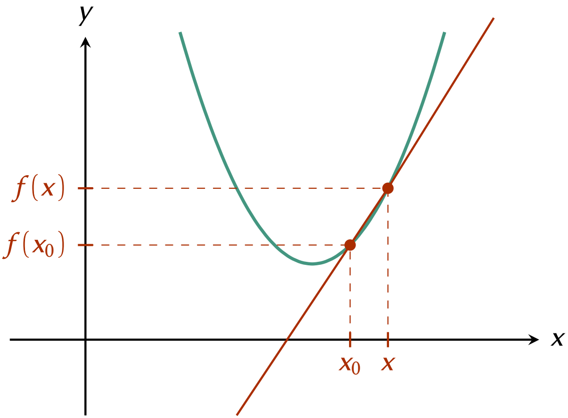

- $f'(x_0) \approx \dfrac{\Delta y}{\Delta x} := \dfrac{f(x) - f(x_0)}{x - x_0}$

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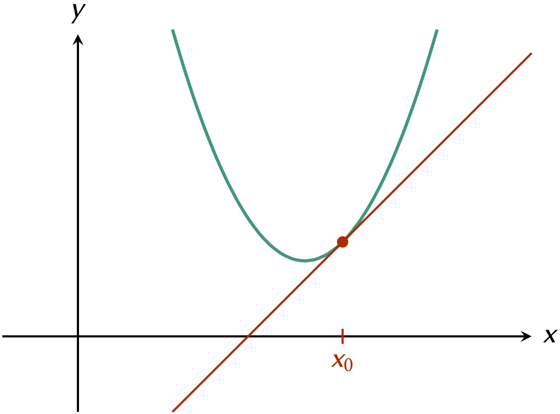

- $f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

- Derivatives of elementary functions

$f(x)$	1	x	x^2	x^p	e^x	$\ln x$	$\sin x$	$\cos x$
$f'(x)$	0	1	$2\,x$	$p\,x^{p-1}$	e^x	$\frac{1}{x}$	$\cos x$	$-\sin x$

Note: $\frac{1}{x^p} = x^{-p}, \quad \sqrt[n]{x} = x^{\frac{1}{n}}$

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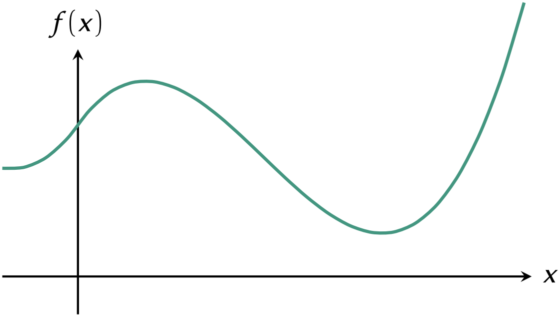
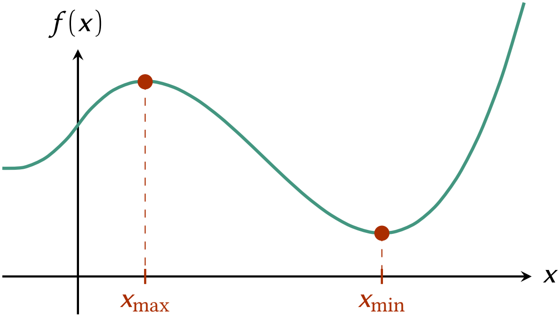
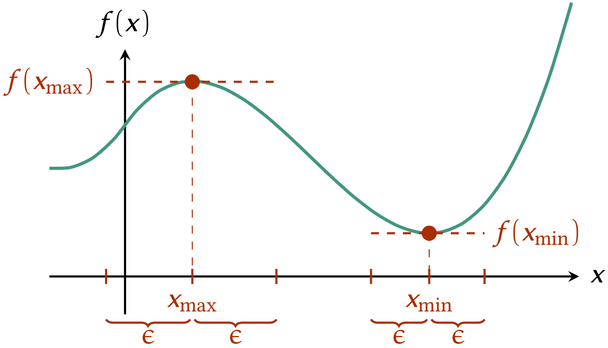
Theorem (Rules for derivatives)

$$(c \cdot f(x))' = c \cdot f'(x)$$
$$(f(x) + g(x))' = f'(x) + g'(x)$$
$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

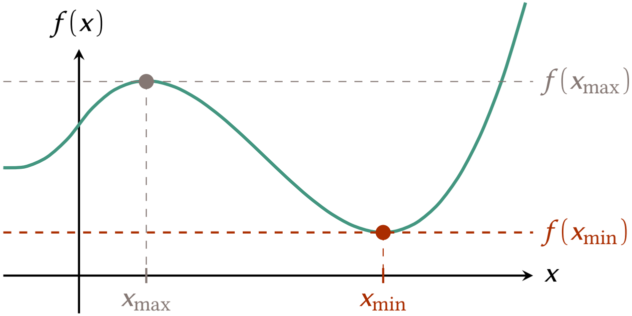

$f(x)$
x

Extrema of functions





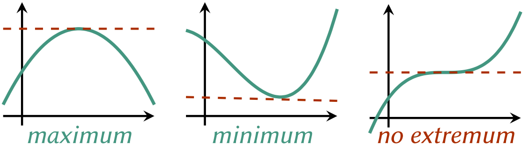
- **Local Maximum:**
  x_{max} with $f(x_{max}) \geq f(x)$
  for all $x \in (x_{max} - \epsilon, x_{max} + \epsilon)$ for some $\epsilon > 0$.
- **Local Minimum:**
  x_{max} with $f(x_{min}) \leq f(x)$
  for all $x \in (x_{min} - \epsilon, x_{min} + \epsilon)$ for some $\epsilon > 0$.



- **Global Maximum:**
  $x_{\max}$ with $f(x_{\max}) \geq f(x)$
  for all $x \in \mathbb{R}$.
- **Global Minimum:**
  $x_{\max}$ with $f(x_{\min}) \leq f(x)$
  for all $x \in \mathbb{R}$.

Computation of Extrema

- Local extrema of differentiable functions
  - **Necessary condition** $f'(x_0) = 0$



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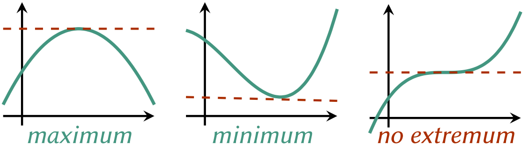
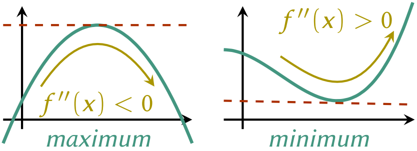


  - **Sufficient condition** $f'(x_0) = 0$ and $f''(x_0) \neq 0$



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- Local extrema of differentiable functions
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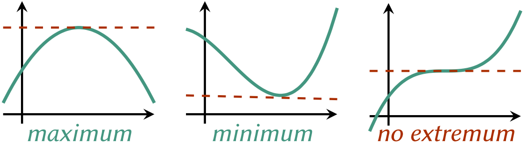

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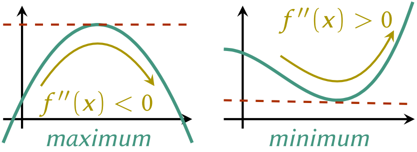
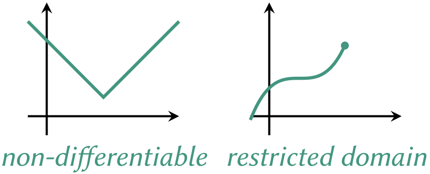


- Special cases:



Motivation

- Sometimes limits are undefined, e.g.

$$\lim_{x \to 0} \frac{sin(x)}{x} = \frac{"0"}{0}$$

- Cases as $\frac{"0"}{0}$ and $\frac{"\infty"}{\infty}$ are not well defined

Theorem

If

$$\lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} g(x) = 0 \text{ or } \lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} g(x) = \pm\infty$$

then

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)}$$

if the latter exists.